A CLASS OF DIRICHLET BOUNDARY VALUE PROBLEMS WHICH ADMIT INFINITELY MANY SOLUTIONS

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Abstract. We indicate sufficient conditions under which the Dirichlet boundary value problem

$$\begin{cases} -\Delta_p u = \lambda u|u|^{q-1} + f(x,u) & \text{in } \Omega \\ u|\partial\Omega = 0 \end{cases}$$

admits infinitely many solutions of negative energy.

Let Ω be a bounded smooth domain in $\mathbf{R}^N$ $(N \geq 3)$ and $0 < q < p-1 < N-1$. Consider the following Dirichlet boundary value problem for the p-Laplacian,

$$(D_\lambda) \quad \begin{cases} -\Delta_p u = \lambda u|u|^{q-1} + f(x,u) & \text{in } \Omega \\ u|\partial\Omega = 0 \end{cases}$$

where $f : \overline{\Omega} \times \mathbf{R} \to \mathbf{R}$, $f = f(x,t)$, is a continuous function which satisfies the following three conditions:

f$_1$) f is odd with respect to the second variable;

f$_2$) f satisfies an estimate of the form

$$|f(x,t)| \leq A_1(x) \cdot |t|^{s_1} + \ldots + A_k(x) \cdot |t|^{s_k}$$

where $s_i \in (q, p-1)$, $A_i \in L^{\alpha_i}(\Omega)$ and $\alpha_i > p^\star/(p^\star - s_i - 1)$ for each i;

f$_3$) There exists $C > 0$, $\alpha \in (0,p)$ and $\mu \in (0, 1/p)$ such that

$$|\mu \cdot f(x,t) \cdot t - F(x,t)| \leq C|t|^\alpha$$

for all x and t.

As usually, $p^\star = \frac{Np}{N-p}$ denotes the critical exponent and

$$F(x,t) = \int_0^t f(x,\xi)d\xi.$$

The problem (D_λ) has been investigated by Bartsch and Willem [2], who were able to prove the existence of infinitely many solutions of negative energy for every $\lambda > 0$, when $p = 2$ and $f(x,t) = t|t|^{r-1}$ with $0 \leq r \leq 2^* - 1$. That gave a positive answer to a problem raised by Ambrosetti, Brezis and Cerami [1]. The aim of our paper is to show that a similar conclusion is valid in the context of p-Laplacian:

Theorem 1. *Under the above conditions on p, q and f, the problem (D_λ) admits infinitely many solutions of negative energy for every $\lambda > 0$.*

A slight modification of our argument yields the existence of infinitely many solutions for (D_λ), even for $\lambda = 0$. However, in this case we are unable to precise the energy sign.

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The energetic functional associated to (D_λ) is $I_\lambda : W_0^{1,p}(\Omega) \to \mathbf{R}$, where

$$I_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{q+1}\int_\Omega |u|^{q+1} dx - \int_\Omega F(x,u)dx.$$

Since I_λ is C^1 and

$$dI_\lambda(u)v = \int_\Omega \nabla u \cdot \nabla v |\nabla u|^{p-2} dx - \lambda \int_\Omega uv|u|^{q-1} dx - \int_\Omega f(x,u)v dx$$

for all $u, v \in W_0^{1,p}(\Omega)$, the proof of Theorem 1 reduces to the fact that I_λ has infinitely many negative critical values. That is done via a well known topological arugment which can be expressed in terms of genus as follows:

Lemma 2. (*See* [4], *Proposition* 9.3). *Suppose that E is a real Banach space and $I \in C^1(E, \mathbf{R})$ is an even functional, bounded from below, which satisfies the Palais-Smale condition. Letting*

$$\Sigma = \{A \subset E \backslash \{0\} \mid A \text{ is closed and symmetric } \}$$

and

$$\Sigma_n = \{A \in \Sigma \mid \gamma(A) \geq n\}, \quad n \in \mathbf{N}$$

each number

$$c_n = \inf_{A \in \Sigma_n} \sup_{u \in A} I(u)$$

is a critical value of I. Moreover, if $c = c_k = \ldots = c_{k+j}$ then

$$\gamma(\{u | I(u) = c \text{ and } I'(u) = 0\}) \geq j + 1.$$

In particular, if $j > 1$, then I admits infinitely many critical points of level c.

We shall show that I_λ fulfills the hypotheses of Lemma 2. Clearly, I_λ is even.

Lemma 3. *The functional I_λ is bounded from below.*

Proof. Because $W_0^{1,p}(\Omega)$ embedds continuously in any $L^r(\Omega)$ with $1 \leq r \leq p^\star$, there exist positive constants $C_0, C_1, \ldots, C_k$ such that

$$I_\lambda(u) \geq \frac{1}{p}\|u\|^p - \frac{\lambda C_0^{q+1}}{q+1}\|u\|^{q+1} - \sum_{i=1}^{k} \frac{C_i^{s_i+1}}{s_i+1}\|A_i\|_{L^{\alpha_i}}\|u\|^{s_i+1}$$

for every $u \in W_0^{1,p}(\Omega)$. The right hand cannot go to $-\infty$, due to the fact that $p > q+1$, $s_1 + 1, \ldots, s_k + 1$. That yields the assertion of our Lemma 3. $\square$

The fact that the critical values c_n are negative (for $n \geq 1$) constitutes Corollary 5 below. For, we need a technical result about the level sets

$$I_\lambda^c = \{u \in W_0^{1,p}(\Omega) | I_\lambda(u) \leq c\}.$$

Lemma 4. *For every $n \in \mathbf{N}^\star$ there exists an $\varepsilon > 0$ such that $\gamma(I_\lambda^{-\varepsilon}) \geq n$.*

Proof. Let E_n be an n-dimensional subspace of $W_0^{1,p}(\Omega)$ and denote by S_r^n the sphere of center 0 and radius $r > 0$ in E_n. Because $W_0^{1,p}(\Omega)$ embedds continuously in any $L^{(s_i+1)\beta_i}(\Omega)$ (where $1/\alpha_i + 1/\beta_i = 1$) we infer the existence of positive constants $C_0, C_1, \ldots, C_k$ such that

$$I_\lambda(\rho u) \leq \frac{1}{p}\rho^p - \frac{\lambda}{q+1}(\rho C_0)^{q+1} + \sum_{i=1}^{k} \frac{(\rho C_i)^{s_i+1}}{s_i+1}\|A_i\|_{L^{\alpha_i}}$$

for every $u \in S_1^n$ and every $\rho > 0$. Consequently, there exist $\rho > 0$ and $\varepsilon > 0$ such that

$$I_\lambda(\rho u) \leq -\varepsilon \text{ for every } u \in S_1^n$$

i.e. $S_\rho^n \subset I_\lambda^{-\varepsilon}$. Because $S_\rho^n \subset \Sigma_n$, we conclude that $\gamma(I_\lambda^{-\varepsilon}) \geq n$. □

Corollary 5. *For every $n \in \mathbf{N}^\star$,*

$$c_n = \inf_{A \in \Sigma_n} \sup_{u \in A} I_\lambda(u) < 0.$$

It remains to prove that I_λ satisfies the Palais - Smale condition. That will be done in several steps, by noticing that dI_λ can be represented as a sum

$$dI_\lambda = D - d\Phi$$

where $D : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ $(1/p + 1/p' = 1)$ is given by

$$D(u)v = \int_\Omega \nabla u \cdot \nabla v |\nabla u|^{p-2} dx$$

and $\Phi : W_0^{1,p}(\Omega) \to \mathbf{R}$ is given by

$$\Phi(u) = \frac{\lambda}{q+1} \int_\Omega |u|^{q+1} dx + \int_\Omega F(x,u) dx.$$

D is an into isomorphism and $d\Phi$ is compact. To prove the later, we shall need the following technical result:

Lemma 6. *Suppose that $(u_n)_n$ is a sequence of elements of $W_0^{1,p}(\Omega)$ such that*

$$u_n \to u \text{ in any } L^{r_i}(\Omega), \text{ where } r_i = \alpha_i s_i p^\star/(\alpha_i(p^\star - 1) - p^\star) \text{ and } i \in \{1, \ldots, k\}$$

and

$$u_n \to u \quad a.e.$$

Then $f(x, u_n) \to f(x,u)$ in $L^{p^\star/(p^\star-1)}(\Omega)$.

Proof. Let $\varphi \in \bigcap_{i=1}^k L^{r_i}(\Omega)$. An easy application of Hölder inequality shows that

$$A_1|\varphi|^{s_1} + \ldots + A_k|\varphi|^{s_k} \in L^{p^\star/(p^\star-1)}(\Omega)$$

which yields the membership of $f(x,\varphi)$ to $L^{p^\star/(p^\star-1)}(\Omega)$. Particularly, this is the case for all $f(x,u_n)$, where $n \in \mathbf{N}^\star$.

We can assume (by passing to a subsequence if nessary) that there exist functions $h_i \in L^{r_i}(\Omega)$ $(1 \leq i \leq k)$ such that $|u_n| \leq h_i$ for every $n \in \mathbf{N}^\star$ and every $i \in \{1, \ldots, k\}$. Then all functions $f(x,u_n)$ are dominated by

$$A_1|h_1|^{s_1} + \ldots + A_k|h_k|^{s_k} \in L^{p^\star/(p^\star-1)}(\Omega)$$

and thus by Dominated Convergence Theorem we can conclude that $f(x,u)$ is in $L^{p^\star/(p^\star-1)}(\Omega)$ too. □

Lemma 7. *The mapping $d\Phi : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is compact.*

Proof. Let $(u_n)_n$ be a bounded sequence of elements of $W_0^{1,p}(\Omega)$. By passing to a subsequence if nessary, we can assume in addition that

$$u_n \to u \text{ in } L^1(\Omega) \text{ and in all spaces } L^{r_i}(\Omega) \ \ (1 \leq i \leq k)$$
$$u_n \to u \ \ a.e.$$

Then in order to end the proof it suffices to show that $d\Phi(u_n) \to d\Phi(u)$. That can be done easily by combining the well known property of continuity of Nemytski operator (see [4], Proposition B1) with the following inequality

$$\begin{aligned}\|d\Phi(u_n) - d\Phi(u)\| \ &\leq \lambda C_1 \left(\int_\Omega \left| u_n |u_n|^{q-1} - u|u|^{q-1} \right|^{1/q} dx \right)^q + \\ &+ C_2 \left(\int_\Omega |f(x,u_n) - f(x,u)|^{p^\star/(p^\star-1)} dx \right)^{1-1/p^\star}.\end{aligned}$$

To prove this inequality notice that $|d\Phi(u_n)v - d\Phi(u)v|$ is majorized, for every $v \in W_0^{1,p}(\Omega)$, by

$$\begin{aligned}&\left| \lambda \int_\Omega \left(\left| u_n v |u_n|^{q-1} - uv|u|^{q-1} \right|^{1/q} \right) dx + \int_\Omega (f(x,u_n)v - f(x,u)v) dx \right| \\ &\leq \lambda \int_\Omega \left| u_n |u_n|^{q-1} - u|u|^{q-1} \right| \cdot |v| dx + \int_\Omega |f(x,u_n) - f(x,u)| \cdot |v| dx \\ &\leq \lambda \left(\int_\Omega \left| u_n |u_n|^{q-1} - u|u|^{q-1} \right|^{1/q} dx \right)^q \cdot \left(\int_\Omega |v|^{1/(1-q)} dx \right)^{1-q} + \\ &\quad + \left(\int_\Omega |f(x,u_n) - f(x,u)|^{p^\star/(p^\star-1)} dx \right)^{1-1/p^\star} \cdot \left(\int_\Omega |v|^{p^\star} dx \right)^{1/p^\star}.\end{aligned}$$

Then the Sobolev's embeddings yield positive constants C_1 and C_2 such that the last estimate can be continued as

$$\begin{aligned}\leq \Bigg[&\lambda C_1 \left(\int_\Omega \left| u_n |u_n|^{q-1} - u|u|^{q-1} \right|^{1/q} dx \right)^q + \\ &+ C_2 \left(\int_\Omega |f(x,u_n) - f(x,u)|^{p^\star/(p^\star-1)} dx \right)^{1-1/p^\star} \Bigg] \cdot \|v\|\end{aligned}$$

and the proof is done. □

Lemma 8. *The functional I_λ satisfies the Palais-Smale condition.*

Proof. Let $(u_n)_n$ be a sequence of elements of $W_0^{1,p}(\Omega)$ such that

$$M = \sup |I_\lambda(u_n)| < \infty \ \text{ and } \ dI_\lambda(u_n) \to 0.$$

We have to show that it contains a converging subsequence. For, notice first that $(u_n)_n$ is necessarily bounded. In fact, for n sufficiently large we have

$$\begin{aligned}M + \mu\|u_n\| &\geq \frac{1}{p} \int_\Omega |\nabla u_n|^p dx - \frac{\lambda}{q+1} \int_\Omega |u_n|^{q+1} dx - \int_\Omega F(x,u_n) dx - \\ &- \mu \int_\Omega |\nabla u_n|^p dx + \lambda\mu \int_\Omega |u_n|^{q+1} dx + \mu \int_\Omega f(x,u_n) u_n dx \\ &= (\frac{1}{p} - \mu) \int_\Omega |\nabla u_n|^p dx - \lambda(\frac{1}{q+1} - \mu) \int_\Omega |u_n|^{q+1} dx + \\ &+ \int_\Omega [\mu f(x,u_n) u_n - F(x,u_n)] dx \\ &\geq (\frac{1}{p} - \mu) \|u_n\|^p - \lambda K_1 (\frac{1}{q+1} - \mu) \|u_n\|^{q+1} - C K_2 \|u_n\|^\alpha,\end{aligned}$$

where K_1 and K_2 are suitable positive constants (and C and μ are given by condition f_3) above).

Now we can apply the compactness of $d\Phi$ to infer the existence of a subsequence $(v_n)_n$ of $(u_n)_n$ such that $(d\Phi(v_n))_n$ is converging, say to w. Because

$$v_n = D^{-1}(dI_\lambda(v_n)) + D^{-1}(d\Phi(v_n))$$

we can conclude that $v_n \to D^{-1}(w)$. $\square$

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